STRUCTURI ALGEBRICE II

OVIDIU FURDUI

Teste Grila de Matematică 2020

$M=[5,7]$.

$x*y = xy-6x-6y+\alpha$

66, 67, 68.

66 M parte stabilă în raport cu operația $*$

$\forall x,y\in M \Rightarrow x*y\in M \Rightarrow$

$5\le x*y\le 7 \quad \forall x,y\in[5,7] \Rightarrow$

$5\le xy-6x-6y+\alpha\le 7$

$5\le (x-6)(y-6)+\alpha-36\le 7 \quad \forall x,y\in[5,7]$

$x-6=a$
$y-6=b$ $\Rightarrow$ $-1\le a,b\le 1$

$5\le ab+\alpha-36\le 7 \Big|_{-6} \Rightarrow$

$-1\le ab+\alpha-42\le 1. \to 41-\alpha\le ab\le 43-\alpha.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow -1 \qquad\qquad \downarrow 1.$

$ab=-1 \Rightarrow \alpha-42\ge 0.$

$ab=1 \Rightarrow \alpha-42\le 0$

$\Rightarrow \boxed{\alpha=42} \Rightarrow$ A

67 In monoidul $(M,*)$ elementul neutru este:

$M\times M\longrightarrow M, \ (x,y)\longrightarrow x*y\in M$

$x * y = xy - 6x - 6y + 42$ $\qquad \alpha = 42$.

$x * e = e * x = x \quad \forall x \in [5,7]$

$x * e = x \cdot e - 6x - 6e + 42 = x$

$xe - 7x - 6e + 42 = 0$.

$(x-6)e - 7(x-6) = 0 \Rightarrow (x-6)(e-7) = 0$.
$\forall x \in [5,7]$

$x = 5$

$\Rightarrow \boxed{e = 7} \Rightarrow$ A

68 $(M, *)$ monoid.
elemente simetrizabile

$x \in [5,7]$ simetrizabil $\exists\, x' \in [5,7]$

$x * x' = x' * x = e = 7$

$x * x' = xx' - 6x - 6x' + 42 = 7 \Big| -6 \Rightarrow$

$xx' - 6x - 6x' + 36 = 1$.

$x(x'-6) - 6(x'-6) = 1 \Rightarrow$

$(x-6)(x'-6) = 1$.

$x, x' \in [5,7] \Rightarrow x-6, x'-6 \in [-1,1]$.

$-1 \le x-6, x'-6 \le 1 \Rightarrow (x-6)(x'-6) = 1 \Rightarrow$

$x-6 = 1$ și $x'-6 = 1$ sau $x-6 = -1$ și $x'-6 = -1$

$x = 7$ sau $x = 5$ $\quad 7' = 7$
$x' = 7$ $\quad\quad x' = 5.$ $\quad 5' = 5$

$7 * 7 = 7 \cdot 7 - \cancel{6 \cdot 7} - 6 \cdot \cancel{7} + \cancel{42} = 49 - 42 = 7 = e$
$7' = 7$

$5 * 5 = 5 \cdot 5 - 6 \cdot 5 - 6 \cdot 5 + 42 = -5 - 30 + 42 = 7$
$5' = 5$

$\{5, 7\} \Rightarrow$ C

69 $\quad \mathbb{Z} * \mathbb{Z}$

$(x, y) * (a, b) = (xa, xb + ya)$

element neutru (e_1, e_2)

$(x, y) * (e_1, e_2) = (e_1, e_2) * (x, y) = (x, y)$
$\forall (x, y) \in \mathbb{Z} \times \mathbb{Z}.$

$*$ - comutativă.

$(x, y) * (e_1, e_2) = (x, y) \quad \forall x, y \in \mathbb{Z}.$

$(xe_1, xe_2 + ye_1) = (x, y) \Rightarrow$

$\begin{cases} xe_1 = x \\ xe_2 + ye_1 = y \end{cases} \Rightarrow \begin{cases} x(e_1 - 1) = 0 & \forall x \in \mathbb{Z} \\ xe_2 + ye_1 = y & \forall x, y \in \mathbb{Z} \end{cases}$

$e_1 = 1 \Rightarrow xe_2 = 0 \Rightarrow e_2 = 0.$
$\forall x \in \mathbb{Z}$

$(e_1, e_2) = (1, 0) \Rightarrow$ B $\quad$ verificare!

70. $x * y = \dfrac{x-y}{1-xy}$; $x, y \in (-1, 1)$

element neutru = ?

$\boxed{x * e = e * x = x}$ $\forall x \in (-1,1)$

$x * e = x \Rightarrow \dfrac{x-e}{1-xe} = x \Rightarrow$

$x - e = x - x^2 e \Rightarrow 0 = e(1-x^2) \Rightarrow e = 0.$

$|x|<1 \quad 1-x^2 \neq 0.$

Verific $e * x = x$. $0 * x = x$.

$0 * x = \dfrac{0-x}{1-0 \cdot x} = \dfrac{-x}{1} = -x \neq x.$

ATENȚIE Legea * nu e comutativă.

$x * y = \dfrac{x-y}{1-xy} \neq y * x = \dfrac{y-x}{1-yx} = -\dfrac{x-y}{1-xy} = -x * y$

$x * y = -\, y * x$

elem. neutru : NU EXISTĂ (B)

71, 72, 73, 74 - clasice
- calcul direct

(189) pe $\mathbb{R}$ $\quad x * y = xy - ax + by$

$a, b \in \mathbb{R}$

$(\mathbb{R}, *)$ monoid $\quad a, b = ?$

monoid $\begin{cases} \text{legea asociativă} \\ \exists \text{ elem. neutru} \end{cases}$

• asociativitate

$$(x * y) * z = x * (y * z) \quad \forall x, y, z \in \mathbb{R}$$

$$(x * y) * z = (xy - ax + by) * z$$

$$= (xy - ax + by) \cdot z - a(xy - ax + by) + bz$$

$$= \boxed{\cancel{xyz} - axz + \cancel{byz} - \cancel{axy} + a^2x - \cancel{aby} + bz}$$

$$x * (y * z) = x * (yz - ay + bz)$$

$$= x(yz - ay + bz) - ax + b(yz - ay + bz)$$

$$= \boxed{\cancel{xyz} - \cancel{axy} + bxz - ax + \cancel{byz} - \cancel{aby} + b^2z}$$

$$-axz + a^2x + bz = bxz - ax + b^2z$$

$$(a+b)xz - (a+a^2)x + (b^2-b)z = 0 \quad \forall x, y, z \in \mathbb{R}$$

$$\Rightarrow \begin{cases} a+b = 0 \\ a+a^2 = 0 \\ b^2 - b = 0 \end{cases}$$

$$(a+b)xz-(a+a^2)x+(b^2-b)z=0. \quad : xz \neq 0.$$

$$a+b-(a+a^2)\cdot\frac{1}{z}+(b^2-b)\cdot\frac{1}{x}=0.$$

$$\forall x, z \neq 0.$$

$z, x \to \infty.$

$z = x = n \to \infty. \Rightarrow \underline{\underline{a+b=0.}}$

$a+b=0.$

$a(a+1)=0 \to a=0$ sau $a=-1$

$b(b-1)=0.$ $\quad b=0$ sau $b=1$

$\boxed{a=0,\ b=0.}$ $\quad \cancel{a=0,\ b=1}$ $\quad \cancel{a=-1,\ b=0}$ $\quad \boxed{a=-1,\ b=1}$

$a+b\neq 0. \quad a+b\neq 0.$

$\underline{a=b=0} \quad x*y = xy$

$\underline{\text{elem. neutru}} \quad x*e = e*x = x \ \forall x\in\mathbb{R}$

$xe = x \Rightarrow x(e-1)=0 \ \forall x\in\mathbb{R}.$ $\quad \boxed{e=1}$

$\underline{a=-1,\ b=1} \Rightarrow x*y = xy+x+y$

$\underline{\text{elem neutru}} \quad x*e=e*x=x \ \forall x\in\mathbb{R}$

$xe+x+e=x \Rightarrow e(x+1)=0 \ \forall x\in\mathbb{R} \ \boxed{e=0}$

$\boxed{C}$

(190) $(\mathbb{C}^*, \cdot)$ și $(\mathbb{R}^*, \cdot)$ grupuri

$a \in \mathbb{R}^*$ și $b \in \mathbb{R}$

$f : \mathbb{C}^* \longrightarrow \mathbb{R}^*$; $f(z) = a|z| + b$.

morfism de grupuri

$f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2)$

$\forall z_1, z_2 \in \mathbb{C}^*$

$a|z_1 z_2| + b = (a|z_1| + b)(a|z_2| + b)$

$\forall z_1, z_2 \in \mathbb{C}^*$

$a|z_1 z_2| + b = a^2 |z_1 z_2| + ab|z_1| + ab|z_2| + b^2$

$(a^2 - a)|z_1 z_2| + ab|z_1| + ab|z_2| + b^2 - b = 0$.

$\forall z_1 z_2 \in \mathbb{C}^*$

$a^2 - a + ab \dfrac{1}{|z_2|} + ab \dfrac{1}{|z_1|} + \dfrac{b^2 - b}{|z_1 z_2|} = 0$

$z_1, z_2 \neq 0$.

$z_1, z_2 = n \longrightarrow \infty$ $\qquad \underline{a^2 - a = 0}$

$ab|z_1| + ab|z_2| + b^2 - b = 0$; $|z_1|$

$ab = 0 \Rightarrow b^2 - b = 0$.

$a^2 - a = 0$. $\Rightarrow a(a-1) = 0$, $\Rightarrow a = 1$

$ab = 0$ $\quad a \neq 0$. $\Rightarrow b = 0$.

$b^2 - b = 0$

$a = 1$ și $b = 0$.

$\boxed{f(z) = |z|}$. $\quad$ c

191 $(\mathbb{Z}[i], \cdot)$ monoid.

$\mathbb{Z}[i] = \{ a + ib ; \ a, b \in \mathbb{Z} \}$

elementele inversabile. ?

elem neutru : 1.

$a + ib \in \mathbb{Z}[i]$ inversabil $\exists\ c + id \in \mathbb{Z}[i]$

a.î. $(a+ib)(c+id) = 1$. $\Rightarrow$

$\overline{(a+ib)(c+id)} = \overline{1} = 1$ $\Rightarrow$ înmulțim

$\Rightarrow (a-ib)(c-id) = 1$

$(a+ib)(a-ib)(c+id)(c-id) = 1$

FORMULĂ $x, y \in \mathbb{R}$ $\boxed{(x+iy)(x-iy) = x^2 + y^2}$

formulă echivalentă $\boxed{z \in \mathbb{C}\ ,\ z\bar{z} = |z|^2}$

$$\left.\begin{aligned}&(a^2+b^2)(c^2+d^2)=1\\ &a,b\in\mathbb{Z} \Rightarrow a^2+b^2\in\mathbb{Z}\\ &c,d\in\mathbb{Z} \Rightarrow c^2+d^2\in\mathbb{Z}\end{aligned}\right\} \Rightarrow$$

$a^2+b^2=1 \Rightarrow a^2=1\ b^2=0$ sau $a^2=0\ b^2=1$

$c^2+d^2=1 \Rightarrow c^2=1\ d^2=0$ sau $c^2=0\ d^2=1$

$a^2=1 \Rightarrow a=\pm 1 \qquad \pm 1$

$b^2=0 \Rightarrow b=0.$

$$\begin{matrix} a^2=0 \\ b^2=1 \end{matrix} \Rightarrow \begin{matrix} a=0 \\ b=\pm 1 \end{matrix} \Rightarrow \pm i$$

$$\begin{matrix} c^2=1 \\ d^2=0 \end{matrix} \Rightarrow \begin{matrix} c=\pm 1 \\ d=0 \end{matrix} \Rightarrow \pm 1.$$

$$\begin{matrix} c^2=0 \\ d^2=1 \end{matrix} \Rightarrow \begin{matrix} c=0 \\ d=\pm 1 \end{matrix} \Rightarrow \pm i$$

elemente inversabile $\{\pm 1, \pm i\}$ B

$= \{-1, 1, -i, i\}$

verificare

192 $\mathbb{Z}$. $x*y = xy + mx + my + a$

$m \in \mathbb{Z}$.

$a = ?$ $(\mathbb{Z}, *)$ - monoid.

$\begin{cases} * \text{ - asociativă} \\ \exists \text{ elem neutru} \end{cases}$

asociativitate

$(x*y)*z = x*(y*z) \quad \forall x, y, z \in \mathbb{Z}$

$(x*y)*z = (xy + mx + my + a) * z$

$= (xy + mx + my + a) z + m(xy + mx + my + a) + mz + a$

$= \boxed{\cancel{xyz} + \cancel{mxz} + \cancel{myz} + az + \cancel{mxy} + m^2x + \cancel{m^2y} + \cancel{am} + mz + \cancel{a}}$

$x*(y*z) = x*(yz + my + mz + a)$

$= \boxed{\cancel{x \cdot yz} + \cancel{mxy} + \cancel{mxz} + ax + mx + \cancel{myz} + \cancel{m^2y} + m^2z + \cancel{ma} + \cancel{a}}$

$az + m^2x + mz = ax + mx + m^2z$

$(a + m - m^2) z + (m^2 - a - m) x = 0 \quad \forall x, z \in \mathbb{Z}$

$(m - m^2 + a)(z - x) = 0. \quad \forall x, z \in \mathbb{Z}$

$m - m^2 + a = 0$ $\boxed{a = m^2 - m}$

$x * y = xy + mx + my + m^2 - m$

<u>elem neutru</u>

$x * e \underset{(1)}{=} e * x = x \quad \forall x \in \mathbb{Z}$

$x * e = xe + mx + me + m^2 - m = x$

$xe + mx - x + m(e + m - 1) = 0$

$x(e + m - 1) + m(e + m - 1) = 0$

$(x + m)(e + m - 1) = 0 \quad \forall x \in \mathbb{Z}$

$\boxed{e = 1 - m}$

$\boxed{E}$

196 $\mathbb{R}$; $x * y = \sqrt[n]{x^n + y^n}$

$\boxed{(\mathbb{R}, *) \text{ grup.} \quad n = ?}$

- $*$ asociativă
- $\exists$ elem neutru
- $\forall$ elem este simetrizabil.

legea asociativă

$(x*y)*z = x*(y*z) \quad \forall x,y,z \in \mathbb{R}$

$(x*y)*z = \sqrt[n]{x^n+y^n+z^n} = x*(y*z)$

elem. neutru

$x*e = e*x = x \quad \forall x \in \mathbb{R}$,

$x*e = x \Rightarrow \sqrt[n]{x^n+e^n} = x \Rightarrow x^n+e^n = x^n$

$\Rightarrow e^n = 0 \Rightarrow \underline{\underline{e=0}}$

Verificare

$x*0 = \sqrt[n]{x^n+0^n} = \sqrt[n]{x^n} = x$.

$\sqrt[n]{x^n} = \begin{cases} x, & n \text{ impar} \\ \pm x, & n \text{ par}. \end{cases} \quad \left(\sqrt[n]{x^n} = |x| = \pm x\right)$

$\Rightarrow \boxed{n \text{ impar}}$ $\boxed{D}$

simetric $x \Rightarrow -x$,

$x*(-x) = (-x)*x = 0$,

$x*(-x) = \sqrt[n]{x^n+(-x)^n} = 0$,

n impar.

Metoda 2:

$f: \mathbb{R} \longrightarrow \mathbb{R}$; $f(x) = \sqrt[n]{x}$

n impar $\Rightarrow$ f bijectivă.

$n = 2k+1$; $k \in \mathbb{N}^*$

$f^{-1}: \mathbb{R} \longrightarrow \mathbb{R}$ $\quad f^{-1}(u) = u^n$;

$$(\mathbb{R}, +) \xrightarrow{f(x) = \sqrt[n]{x}} (\mathbb{R}, *)$$

$(\mathbb{R}, +)$ grup.

$f(x) = \sqrt[n]{x}$ - bijectie $\Rightarrow$

induce pe $\mathbb{R}$ operatia "$*$"

definită prin

$f(x+y) = \underbrace{f(x)}_{u} * \underbrace{f(y)}_{v}$ $\quad f(x) = u \Rightarrow x = f^{-1}(u)$

$u * v = f(f^{-1}(u) + f^{-1}(v))$

$= f(u^n + v^n) = \sqrt[n]{u^n + v^n}$

$u * v = \sqrt[n]{u^n + v^n}$ $\quad u, v \in \mathbb{R}$.

$(\mathbb{R}, +) \cong (\mathbb{R}, *)$ $\quad u * v = \sqrt[n]{u^n + v^n}$;

197 $(M_2(\mathbb{Z}), \cdot)$ monoid.

elemente inversabile

$A \in M_2(\mathbb{Z})$ inversabil $\Rightarrow \exists B \in M_2(\mathbb{Z})$

$AB = BA = I_2$.

$AB = I_2 \Rightarrow \det A \det B = 1 \Rightarrow \underline{\underline{\det A = \pm 1}}$.

$\det A, \det B \in \mathbb{Z}$.

<u>Observatie</u>

$\underline{A \in M_2(\mathbb{C})}$

$A^2 - (\operatorname{tr} A) A + (\det A) I_2 = O_2$.

$\det A \neq 0 \Rightarrow A^2 - (\operatorname{tr} A) A = -(\det A) I_2$.

$\Rightarrow A(A - (\operatorname{tr} A) I_2) = -(\det A) I_2 \quad : -\det A$

$$A \underbrace{\left(\frac{A - (\operatorname{tr} A) I_2}{-\det A} \right)}_{A^{-1}} = I_2 \Rightarrow$$

$$\boxed{A^{-1} = \frac{1}{\det A} \left[(\operatorname{tr} A) I_2 - A \right]}$$

E

$\det A = \pm 1 \Rightarrow A^{-1} = \pm 1 \left[(\operatorname{tr} A) I_2 - A \right]$.

232 $f(x,y)=\dfrac{ax+by}{1+xy}$, $a,b\in\mathbb{R}$

lege de compozitie pe $(-1,1)$ dacă

(*) $\boxed{-1<\dfrac{ax+by}{1+xy}<1 \quad \forall x,y\in(-1,1)}$

$x=0$. $-1<by<1 \quad \forall y\in(-1,1)$

$y\to 1 \Rightarrow -1\le b\le 1 \Rightarrow$

$b\in[-1,1]$.

$y=0$. $-1<ax<1 \quad \forall x\in(-1,1)$

$x\to 1 \quad -1\le a\le 1 \quad a\in[-1,1]$.

$y=-x \quad -1<\dfrac{(a-b)x}{1-x^2}<1 \quad \forall x\in(-1,1)$.

$-(1-x^2)<(a-b)x<1-x^2 \quad \forall x\in(-1,1)$

$x\to 1 \quad 0\le a-b\le 0 \Rightarrow a=b$

$a=b\in[-1,1]$. $\to$ $\boxed{\Delta}$

<u>Verificare</u> $-1<a\cdot\dfrac{x+y}{1+xy}<1 \quad \forall x\in(-1,1)$

$|a|\le 1$

$$\left|a\cdot\frac{x+y}{1+xy}\right| = |a|\left|\frac{x+y}{1+xy}\right| \le \left|\frac{x+y}{1+xy}\right| < 1$$

pt că

$$-1 < \frac{x+y}{1+xy} < 1 \quad \forall x,y\in(-1,1)$$

vezi lecția I.

sau calcul direct

$$-1-xy < x+y < 1+xy \quad \forall x,y\in(-1,1).$$

$$x+y<1+xy \Rightarrow 1+xy-x-y>0. \Rightarrow (1-x)(1-y)>0.$$

$$-1-xy < x+y \Rightarrow x+y+1+xy = (1+x)(1+y)>0.$$